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# Exploring Two Methods of Partial Fraction Decomposition on Students' Performance

**Jong Li Ling[1] and Kuan Siaw Vui [2]**

[1]*Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Cawangan Sarawak, Malaysia*
[2]*Faculty of Business Management, Universiti Teknologi MARA Cawangan Sarawak, Malaysia*
**Corresponding author: [1]lljong@uitm.edu.my*

## ABSTRACT

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*Different partial fraction decomposition (PFD) methods may drive students to explore and understand partial fractions and thus, improve their mastery in PFD performance. Hence, this study explored the effectiveness of using two different methods, namely the improved version of the Heaviside method and the undetermined coefficients method, in performing PFD of the proper partial fraction. Literature showed that most of the instructors employed the undetermined coefficients method, and little is known about the effectiveness of employing other methods on students' performance. This study used a quasi-experimental approach with a pre-test and post-test interval. Purposive sampling was employed as all the participants are from science stream, have completed Calculus I course, and learnt PFD. A total of 148 undergraduates from two faculties of a Malaysian public university were purposefully chosen for this study. The pre-test and post-test scores of PFD for three categories of factors in the denominator using the two methods were collected. Then, the statistical results of pre-test and post-test were examined using IBM SPSS 21. The mean scores of the tests were analysed using paired sample t-tests and analysis of covariance. The findings revealed that students who used the improved version of the Heaviside method outperformed those who used the undetermined coefficients method in performing the PFD of proper rational functions for distinct linear factor and irreducible quadratic factor in the denominators. However, the performance for both methods was insignificantly different for solving PFD of proper rational functions concerning repeated linear factor in the denominator. This study provides valuable insights into the choice of PFD methods employed by instructors in bringing out the best in students' performance.*

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## 1. INTRODUCTION

Partial fraction decomposition (PFD) is the process of decomposing a complex rational fraction into the sum of simple rational fractions. PFD is thus the reverse of the summation of simple rational fractions. In learning elementary integral calculus, PFD is the initial step of computation before integrating. It is usually easier to integrate simple rational fractions than to integrate complex rational functions. Performing the PFD computation effectively with high numerical accuracy is often the primary concern in PFD learning. Though numerous algorithms or approaches to decompose certain types of rational functions into partial fractions are available, not all of them are suitable for manual calculation. Fundamental knowledge of certain methods used to complete PFD computation is also needed. For instance, the PFD coefficient can be found by using Taylor polynomial computation (Kwang & Xin, 2018) but to the knowledge of the researchers in this study, students must be well-versed in the divide-and-conquer method to perform this computation effectively. In addition, the PFD coefficient can also be found by employing repeated synthetic division (Kim & Lee, 2016). However, to our knowledge and based on our observations, students need to apply repeated synthetic division. Furthermore, the PFD coefficient can be found using the differentiation method (Özyapici & Pintea, 2012) but students must have a strong basic knowledge of differentiation to conduct the computation. Although Wang (2007) proposed a set of PFD formulas that can be used for immediate integration, students must have superior memorization reasoning to memorize the formulae needed to perform the computation.

The theoretical and empirical literature review shows that students are often introduced to use the undetermined PFD method in solving PFD coefficients at schools and even higher learning institutions. (e.g., William, 2018; Manoj, Ashvini, & Hole, 2020; Kwang & Xin, 2018). The undetermined coefficients PFD method largely emphasizes the use of the algebra approach for solving PFD coefficients. However, students who are not proficient in applying basic algebraic concepts for solving PFD coefficients will eventually end up with a poor performance in PFD. Furthermore, it has an impact on integrating proper rational fractions when students make mistakes or use incorrect PFD coefficients. Hence, students tend to lose marks in this whole process of performing PFD and integrating proper rational fractions which will ultimately affect their overall performance.

The concerns and awareness of the limitations discussed above, specifically in employing undetermined coefficients PFD method to solve PFD coefficients have prompted the researchers of this study to look for other PFD methods that can lead to optimal student performance. Therefore, a complete computation of the improved Heaviside PFD method introduced by Man (2012) was explored in this study. This method uses the formulation of simple polynomial division and the substitution concept to obtain PFD coefficients. The fewer steps required in this method help reduce students' errors in computation and prompt them to obtain accurate solutions as compared to using the undetermined coefficients PFD method. To obtain insights into the two PFD methods chosen for this study, we explore the effectiveness of these two PFD methods on students' PFD performance of the proper rational function.

## 2. LITERATURE REVIEW

Several computation methods of decomposing a rational function into partial fraction have been broadly employed in the application of calculus, differential equations, control theory and some areas of pure or applied mathematics (Kwang & Xin, 2018; Manoj et al., 2020; Kim & Lee, 2016; Ma et al., 2014; Bradley & Cook, 2012; Man, 2012; Özyapici & Pintea, 2012). However, it is observed that two PFD coefficients computation methods that are more commonly used, namely the undetermined coefficients method and the cover-up method. (e.g., William, 2018; Kim & Lee, 2016; Ma et al., 2014; Man, 2012). According to Linner (1974, cited in Ma et al., 2014), the well-known cover-up method always serves as a basis for other PFD methods and provides a compact solution to PFD problems. This, however, has a limitation when it comes to the evaluation of high-order poles in high-order polynomials as it could result in huge numerical errors when the successive differentials procedures increase (Ma et al., 2014; Kwang & Xin, 2018). Another standard PFD method, namely the undetermined coefficients method, requires the construction of a system of equations by matching up the variables after removing the fractions form from the combination of partial fractions and a proper rational function using the least common denominator procedure and resolving of the resultant system of equations to obtain PFD coefficients. It can be a very lengthy, complicated, and inconvenient computation when decomposing more than two partial fractions (Wang, 2007; Gupta, 2011, Man, 2012). Therefore, there is a higher possibility for students to make more arithmetic mistakes in this whole process of computation.

The extant literature shows that many students have difficulties solving questions that are associated with the concepts of fractions and algebraic expressions. Titus' (2010) study reported 35% to 42% of the college students enrolled in development mathematics course committed error patterns in the real number computations because most of the students have an unclear understanding of signed number arithmetic, fractions, distributive property, as well as exponential errors. Moreover, Brown and Quinn's research (2006) discovered that more than half of the 143 ninth graders who enrolled in an elementary algebra course at an upper-middle-class school showed a lack of experience and had low proficiency in both fraction concepts and computations. In addition, Bentley and Bossé's (2018) study supported Gabriel et al.'s (2013) finding that college students committed mistakes in wrongly applying fraction operations, as seen in elementary students' misunderstandings and misconceptions. Hanson and Hogan (2000) who examined the computational estimation skills of 77 college students majoring in a variety of disciplines discovered that many students struggled with the process of obtaining common denominators. They also highlighted that few students in the lower performing groups, added or subtracted the numerators and denominators but failed to find common denominators. Furthermore, Steen (2007) emphasized that even adults were found confused if a problem requires anything in the simplest of fractions. Considering the above findings, students' difficulties with fraction concepts are found to be partly responsible for failure in finding PFD coefficients using undetermined coefficients method computation. Hence, many instructors seek alternative methods that could increase the accessibility of the PFD method for students who are weak in concepts of fractions.

Another error pattern, namely difficulties with algebraic equations and arithmetical computation in schools has also been well-documented. The difficulties are related to the inability to see the algebraic structures of the tasks, inadequate conceptual knowledge of the problem, a lack of manipulative expression skills, calculation mistakes, and technical errors (Taban & Cadorna, 2018). In addition, algebra's structure sense is said to be a part of students' difficulties. The difficulties in structure sense include using arithmetical operations in numerical and algebraic expressions, understanding the notion of variables, algebraic expressions, as well as determining the meanings of the equal sign and mathematization (Jupri et al., 2014; Hoch & Dreyfus, 2010). It is also reported that students with high-performance mathematics in secondary schools also had difficulty with algebraic manipulation. They struggled to formulate equations by manipulating correct algebraic expressions; they had weak arithmetic skills; and they made arithmetic errors which caused them to make algebraic errors (Novotna & Hoch, 2008). Another finding shows that students find it difficult to apply previously learnt algebraic techniques (Matzin & Shahrill, 2015). The in-depth analysis on school-aged students' errors in algebra problem solving conducted by Booth and colleagues (2014) reveals six common errors made: variable errors, negative sign errors, equality or inequality errors, operation errors, mathematical properties errors, and fraction errors. Moreover, Ashlock (2010) in his analysis of error patterns made by students discovered that, school-aged students often have misconceptions and make procedural errors in both mathematical operations and methods of computations. These error analyses highlight the most crucial computational mistakes committed by students prior to obtaining the final PFD coefficients when undetermined coefficients method computation was being carried out.

Concerning the above discussions, many PFD methods were proposed to complement the undetermined coefficients methods commonly and widely employed by instructors. Some of the methods are found to perform better than undetermined coefficients methods under specific conditions. For example, some methods are more suitable for small-scale problems, but they may become complicated when used for large-scale problems. In Man's (2007) research, he proposed a Heaviside's cover-up method, which requires simple substitutions to find partial fraction coefficient with single poles and apply successive differentiation for multiple poles. Man (2012) subsequently proposed an improved version of Heaviside's approach to compute partial fraction coefficients by using simple substitutions and polynomial divisions. This method does not require solving the complex roots of the quadratic polynomial, differentiation, or the solution of a system of linear equations for the PFD of a proper rational function. Its simplicity and applicability in applied and engineering mathematics as recommended by several researchers (e.g., William, 2018; Manoj et al., 2020; Man, 2012) to employ this improved method in teaching integrals of proper rational functions have compelled the researchers of this current study to explore the potential application of this method on teaching undergraduate students as an alternative method to the undetermined coefficients method in finding the PFD coefficients.

To further examine students' understanding in applying partial fraction decomposition method, the effectiveness of applying the improved version of the Heaviside PFD method and the undetermined coefficients PFD method on their PFD performance is explored. Thus, the research question of this study is: Which application of PFD method (the improved version of Heaviside PFD method or the undetermined coefficients PFD method) improve students' performance? The following Null Hypotheses were developed to answer the research question:

1. The improved version of Heaviside method has no effect on students' PFD performance.

2. The undetermined coefficients method has no effect on students' PFD performance.

3. There is no significant difference between students' PFD performance taught with improved version of the Heaviside method and those taught with undetermined coefficients method.

### *2.1 Partial Fraction Decomposition*

A brief description of a partial fraction decomposition is presented in the next page:

Assume that G is a constant field comprises two polynomials, $\mathrm{W}(x)$ and $\mathrm{S}(x)$. A proper rational function is $\mathrm{G}(x)=\frac{\mathrm{W}(x)}{\mathrm{S}(x)}$, where the degree of $\mathrm{W}(x)$ is lower than the degree of $\mathrm{S}(x)$ and $\mathrm{S}(x)=\prod_{i=1,l=1}^{i=m,l=n}\left(x-\mathrm{a}_i\right)^{j_i}\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^{k_l}$, $\mathrm{a}_i,\mathrm{b}_l,\mathrm{c}_l$ are constants with $\mathrm{b}_l{}^2-4\mathrm{c}_l<0$, $\mathrm{S}(x)$ never be 0 and belongs to G, and $i,j,k,l,n,m$ are positive integers.

A partial fraction decomposition of G($x$) is:
$\mathrm{G}(x)=\sum_{i=1}^{m}\sum_{t=1}^{j_i}\frac{\mathrm{A}_{it}}{\left(x-\mathrm{a}_i\right)^t}+\sum_{l=1}^{n}\sum_{t=1}^{k_l}\frac{\mathrm{B}_{lt}\,x+\mathrm{C}_{lt}}{\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^t}$, where $\mathrm{A}_{it},\mathrm{B}_{lt},\mathrm{C}_{lt}$ are coefficients constants with *t* representing positive integers.

Two methods of PFD were used in this study to compute the unknown coefficients $\mathrm{A}_{it},\mathrm{B}_{lt},\mathrm{C}_{lt}$ and followed the procedure as shown below:

#### *2.1.1 The Improved Version of Heaviside Method*

For distinct and repeated linear polynomial denominator, assume that $\mathrm{B}_{lt}$ and $\mathrm{C}_{lt}$ are zeros, and multiplying the equation of G($x$) with $\left(x-\mathrm{a}_i\right)^t$, and replacing $x$ with $\mathrm{a}_i$ to get coefficient -*i,t* of A, polynomial division, $\mathrm{A}_{it}=\frac{\mathrm{W}(x)}{\mathrm{S}(x)}\left(x-\mathrm{a}_i\right)^t\,|_{x=\mathrm{a}_i}$ is obtained. In order to obtain the next coefficient -*i,t-j<sub>i</sub>* of A in the polynomial division, the known partial fractions are subtracting from F($x$), $\mathrm{A}_{i(t-j_i)}=\left(\frac{\mathrm{W}(x)}{\mathrm{S}(x)}-\sum_{k=0}^{j_i-1}\frac{\mathrm{A}_{i,j_i-k}}{\left(x-\mathrm{a}_i\right)^{j_i-k}}\right)\left(x-\mathrm{a}_i\right)^{t-j_i}\,|_{x=\mathrm{a}_i}$ or using the straightforward process, $\mathrm{A}_{i(t-j_i)}=\frac{\mathrm{W}_{j_i}(x)}{\mathrm{S}_{j_i}(x)}\left(x-\mathrm{a}_i\right)^{t-j_i}\,|_{x=\mathrm{a}_i}$. This process would be progressing until all the unknown coefficients $\mathrm{A}_{it}$ are found.

For irreducible quadratic polynomial denominator, assume that $\mathrm{A}_{it}$ is zero and multiplying the equation of G($x$) with $\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^t$ and modifying the numerator and denominator for the purpose of replacing $x^2$ with $\mathrm{b}_l\,x+\mathrm{c}_l$ to obtain coefficient -*i,t* of B and C in polynomial division, $\mathrm{B}_{it}\,x+\mathrm{C}_{it}=\frac{\mathrm{W}(x)}{\mathrm{S}(x)}\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^t\Big|_{x^2=-\mathrm{b}_l\,x-\mathrm{c}_l}$. Repeat the same process that described above to find all the unknown coefficients, $\mathrm{B}_{lt}$ and $\mathrm{C}_{lt}$ by the polynomial division formula,

$$\mathrm{B}_{i(t-k_l)}\,x+\mathrm{C}_{i(t-k_l)}=\left(\frac{\mathrm{W}(x)}{\mathrm{S}(x)}-\sum_{k=0}^{k_l-1}\frac{\mathrm{B}_{i,j_i-k}\,x+\mathrm{C}_{i,k_l-k}}{\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^{k_l-k}}\right)\left(x^2+\mathrm{b}_l\,x+\mathrm{c}_l\right)^t\Bigg|_{x^2=-\mathrm{b}_l\,x-\mathrm{c}_l}.$$

### *2.1.2 The Undetermined Coefficients Method*

By multiplying the equation of G(x) with the common denominator to obtain a polynomial equation with coefficients which are linear expressions of the constants A, B, and C. The coefficients of the same terms are equalized to form a system of linear equations since both sides of polynomials are equal only on condition that their corresponding coefficients are equal. The system of linear equations is then solved to find all unknown coefficients $\mathrm{A}_{11,\ldots,}\mathrm{A}_{it},\mathrm{B}_{11,\ldots,}\mathrm{B}_{lt},\mathrm{C}_{11,\ldots\ldots,}\mathrm{C}_{lt}$.

The three questions below were used in this study to describe the ways to obtain answers to the questions using two methods:

#### *2.1.2.1 Question 1*

Find the partial fraction expansion of the proper rational function for distinct linear polynomial denominator, $\mathrm{G}(x)=\dfrac{1-2x}{6x^2+x-1}$.

*Solution:*

The PFD expressed as $\dfrac{\mathrm{A}_1}{3x-1}+\dfrac{\mathrm{A}_2}{2x+1}$ where $A_1$ and $A_2$ are unknown coefficients to be determined.

*A. The Improved Version of Heaviside Method*

Using polynomial division formula, the following steps are performed:

$$\mathrm{A}_1\,|_{x=\frac{1}{3}}=\frac{(1-2x)(3x-1)}{(2x+1)(3x-1)}=\frac{1-2\left(\frac{1}{3}\right)}{2\left(\frac{1}{3}\right)+1}=\frac{1}{5}$$

$$\mathrm{A}_2\,|_{x=\frac{-1}{2}}=\frac{(1-2x)(2x+1)}{(3x-1)(2x+1)}=\frac{1-2\left(\frac{-1}{2}\right)}{3\left(\frac{-1}{2}\right)-1}=\frac{-4}{5}$$

*B. The Undetermined Coefficients Method*

Multiplying G($x$) with the common denominator as follows:

$$\frac{(1-2x)(2x+1)(3x-1)}{(2x+1)(3x-1)}=\frac{\mathrm{A}_1}{(3x-1)}(2x+1)(3x-1)+\frac{\mathrm{A}_2}{(2x+1)}(2x+1)(3x-1)$$

To obtain polynomial equation,
$1-2x=(2x+1)\mathrm{A}_1+(3x-1)\mathrm{A}_2$

Hence, the coefficients of the same terms to form a system of linear equations is equalized.
constant, $1=\mathrm{A}_1-\mathrm{A}_2$
$x$ term, $-2=(2)\mathrm{A}_1+(3)\mathrm{A}_2$

Lastly, the above system of linear equations to find $A_1$ and $A_2$, are solved.

Thus, partial fraction of $\mathrm{F}(x)=\dfrac{1}{5(3x-1)}-\dfrac{4}{5(2x+1)}$.

***2.1.2.2 Question 2***

Find the partial fraction expansion of the proper rational function for repeated linear polynomial denominator, $\mathrm{G}(x)=\dfrac{3+4x}{(x)(x+5)^2}$.

*Solution:*

The PFD expressed as $\dfrac{\mathrm{A}_1}{x}+\dfrac{\mathrm{A}_2}{x+5}+\dfrac{\mathrm{A}_3}{(x+5)^2}$ where $A_1$, $A_2$ and $A_3$ are unknown coefficients to be determined.

*A. The Improved Version of Heaviside Method*
Using polynomial division formula to find $A_1$, $A_2$ and $A_3$, the following steps are performed:

$$\mathrm{A}_1\,|_{x=0}=\frac{(3+4x)(x)}{(x)(x+5)^2}=\frac{(3+4(0))}{(0+5)^2}=\frac{3}{25}$$

$$\mathrm{A}_3\,|_{x=-5}=\frac{(3+4x)(x+5)^2}{(x)(x+5)^2}=\frac{(3+4x)}{(x)}=\frac{(3+4(-5))}{(-5)}=\frac{17}{5}$$

Thus, the value of a and c are substituted into G($x$), the following steps are performed:

$$\frac{(3+4x)}{(x)(x+5)^2}=\frac{3}{25x}+\frac{\mathrm{A}_2}{x+5}+\frac{17}{5(x+5)^2}$$

To solve b, $x$ is replaced with one,

$$\frac{(3+4(1))}{(1)(1+5)^2}=\frac{3}{25}+\frac{A_2}{1+5}+\frac{17}{5(1+5)^2}$$

$$A_2=\frac{-3}{25}$$

*B. The Undetermined Coefficients Method*
Multiplying G($x$) with the common denominator as follows:

$$\frac{(3+4x)(x)(x+5)^2}{(x)(x+5)^2}=\frac{A_1}{(x)}(x)(x+5)^2+\frac{A_2}{(x+5)}(x)(x+5)^2+\frac{A_3}{(x+5)^2}(x)(x+5)^2$$

To obtain polynomial equation:

$$(3+4x)=A_1(x+5)^2+A_2(x)(x+5)+A_3(x)$$

The coefficients of the same terms to form a system of linear equations are then equalized.
constant, $3=25A_1$
$x$ term, $4=(10)A_1+(5)A_2+A_3$
$x^2$ term, $0=(1)A_1+(1)A_2$

Lastly, the system of linear equations to find $A_1$, $A_2$ and $A_3$ are solved.

Thus, partial fraction of $F(x)=\frac{3}{25x}-\frac{3}{25(x+5)}+\frac{17}{5(x+5)^2}$.

### *2.1.2.3 Question 3*

Find the partial fraction expansion of the proper rational function for irreducible quadratic polynomial denominator, $G(x)=\frac{2-5x}{(x)(x^2+16)}$.

*Solution:*

The PFD expressed as $\frac{A_1}{x}+\frac{A_2x+A_3}{x^2+16}$ where $A_1$, $A_2$ and $A_3$ are unknown coefficients to be determined.

*A. The Improved Version of Heaviside Method*
Using polynomial division formula to find $A_1$, and by adding $x$ to the polynomial division formula to find $A_2$ and $A_3$ as follows:

$$A_1|_{x=0}=\frac{(2-5x)(x)}{(x)(x^2+16)}=\frac{(2-5(0))}{(0^2+16)}=\frac{2}{16}$$

$$\mathrm{A}_2 x+\mathrm{A}_3 \Big|_{x^2=-16}=\frac{(2-5x)\left(x^2+16\right)}{(x)\left(x^2+16\right)}=\frac{(2-5x)(x)}{(x)(x)}=\frac{(2x+16(5))}{(-16)}=\frac{-2x}{16}-\frac{80}{16}$$

*B. The Undetermined Coefficients Method*
Multiplying G($x$) with the common denominator as follows:

$$\frac{(2-5x)(x)\left(x^2+16\right)}{(x)\left(x^2+16\right)}=\frac{\mathrm{A}_1}{(x)}(x)\left(x^2+16\right)+\frac{\mathrm{A}_2 x+\mathrm{A}_3}{\left(x^2+16\right)}(x)\left(x^2+16\right)$$

To obtain polynomial equation:

$$(2-5x)=\left(x^2+16\right)\mathrm{A}_1+(x)\left(\mathrm{A}_2 x+\mathrm{A}_3\right)$$

The coefficients of the same terms to form a system of linear equations are equalized.
constant, $2=16\mathrm{A}_1$
$x$ term, $-5=\mathrm{A}_3$
$x^2$ term, $0=(1)\mathrm{A}_1+(1)\mathrm{A}_2$

Lastly, the system of linear equations to find $A_1$, $A_2$ and $A_3$ are solved.

Thus, partial fraction of $\mathrm{F}(x)=\frac{2}{16x}+\frac{-2x-80}{16\left(x^2+16\right)}$.

## 3. METHODOLOGY

### *3.1 Participants*

In this study, purposive sampling was employed. The participants in this study were all purposefully chosen since they all had a similar foundation in science, had completed Calculus I in the previous semester, and were taught PFD as part of their course. This type of sampling will yield useful information about the expected outcome of the study. The participants of this study comprised 148 undergraduate students from the Faculty of Civil Engineering and the Faculty of Applied Sciences in the Universiti Teknologi MARA Sarawak Branch. From the total of 148 students in this study, 72 were enrolled in Calculus II for Engineers course during semester two while 66 in the Foundation of Applied Mathematics course at semester three. This study was conducted during the academic year 2020-2021.

### *3.2 Instrumentation*

For this study, an experimental design was adopted, and pre-test and post-test were administered. These tests were conducted to evaluate the students' performance in solving questions testing their knowledge on partial fraction decomposition using two different methods. The results obtained provide a better understanding of how well students perform partial fraction decomposition using both the undetermined coefficients method and the improved version of Heaviside's cover-up method.

The pre and post-tests contained 3 questions. The questions were adopted from the university's Item Bank System (IBS). An item bank is a computerized collection of test items. The assessment specification table (JSU) was used to design test items, which complied with the Ministry of Education's requirements and university's guidelines. The JSU was developed to ensure test items were of high quality, valid, reliable, fair, and consistent. For efficient management procedure, Academic Affairs designated an experienced Resource Person (RP) to examine and approve test items and scoring rubrics that met JSU standards. (Raus et al., 2014; Vaibhav & Arvind, 2013; Syahidah & Nazlia, 2012). Minor adjustments were made based on the review feedback obtained from two experienced lecturers who are experts in this subject as well as guidelines elicited from the literature (Betsy, Kasturi, Chiang, & Goh, 2015). The questions were on proper rational function. The corresponding category of proper rational functions included distinct linear factors in the denominator, repeated linear factors in the denominator and irreducible quadratic factors in the denominator. The marks allocated to each question are ranged from 0 to 10, with 0 for incorrect answers and 10 for correct answers.

All the students were taught about two PFD methods by the same instructor. Students took a pre-test after completing the 6-hour instruction. Students were asked to use the undetermined coefficients method and the improved Heaviside's cover-up method to decompose the proper rational functions with different categories of denominators. The duration of the test is 60 minutes. The pre-test and post-test were administered one month apart to ensure that students have acquired and mastered the knowledge in applying the two different methods for solving partial fraction decomposition problems. Each question was marked using the modified scoring rubric for undetermined coefficients PFD method, which was adapted from IBS.

## 4. DATA ANALYSIS AND RESULTS

The statistical results of pre-test and post-test on solving PFD by employing the two methods were analysed using IBM SPSS 21 with respect to the three categories of factors in the denominator under study. The study omitted about 20% of the total results as some students were absent from either one of the tests administered. A total of 105 results for distinct linear factor and 106 results for irreducible quadratic factor and repeated linear factor were examined. The normality of difference means scores obtained between the pre-tests and post-tests using the two different methods were tested using the Kolmogorov–Smirnov(K-S) test, z test for kurtosis(K) and skewness (SK) coefficients at 0.05 level of significance. All results of the K-S test for means scores obtained were statistically significant.

However, in Table 1, z tests of means scores for K and SK was within $\pm$ 3 indicating that the distribution of means scores obtained were approximately normal (Orcan, 2020; Mishra et al., 2019; Kim, 2013). Besides, paired sample t-tests were conducted to evaluate the differences in students' performance in finding the partial fraction decomposition of proper rational function using the two methods at a 0.05 level of significance. In addition, an analysis of covariance (ANCOVA) was used to compare the efficacy of applying the improved Heaviside's cover-up method and undetermined coefficients method on finding PFD performance while controlling for the initial test at a 0.05 level of significance. For ANCOVA, the homogeneity of variations in the PFD performance was checked by performing Levene's test.

Table 1. Z Test of Mean Scores for Skewness and Kurtosis

| | | Heaviside's Cover-Up Method | | | Undetermined Coefficients Method | | |
|---|---|---|---|---|---|---|---|
| | | | Skewness | | | Kurtosis | |
| **Partial fraction** | **Type of test** | **Value** | **Standard Error** | **z** | **Value** | **Standard Error** | **z** |
| Distinct linear factor | Pre | -1.13 | 0.38 | -2.97 | -0.61 | 0.24 | -2.59 |
| | Post | 0.03 | 0.47 | 0.06 | -0.99 | 0.47 | -2.13 |
| Repeated linear factor | Pre | 1.19 | 0.40 | 2.98 | 0.43 | 0.24 | 1.81 |
| | Post | 1.28 | 0.47 | 2.76 | -0.98 | 0.47 | -2.10 |
| Irreducible quadratic factor | Pre | 0.32 | 0.24 | 1.37 | 0.09 | 0.24 | 0.37 |
| | Post | -0.61 | 0.47 | -1.31 | -0.56 | 0.47 | -1.20 |

Table 2 shows PFD performance using the improved Heaviside's cover-up method. The mean pre-test scores for repeated linear factor, irreducible quadratic factor, and distinct linear factor were 2.49, 2.82, and 6.10, respectively while the mean post-test scores were 4.12, 6.84, and 8.59, respectively. The results of the means difference between the pre-test and post-test revealed a statistically significant improvement in finding the partial fraction decomposition of proper rational function for the three distinct factors in the denominators. Students showed better performance for irreducible quadratic factor compared to distinct linear factor and repeated linear factor. The effect sizes were 0.74, 0.66 and 1.32 for distinct linear factor [$t(104) = 8.35$, $p < .05$; 95% CI (1.9, 3.09)], repeated linear factor [$t(105) = 8.94$, $p < .05$; 95% CI (1.27, 1.99)], and irreducible quadratic factor [$t(105) = 14.61$, $p < .05$; 95% CI (3.47, 4.56)], respectively. These effect sizes indicated moderate-to-large effects (Cohen, 1988; Kraft, 2020; Mathew, 2019; Nicolas, 2017). It reflects that a well-designed Heaviside's cover-up method will increase student's ability to find partial fraction decomposition of a proper rational function therefore improving their performance.

Table 2. Students' Performance in Pre-Test and Post-Test Using the Improved Heaviside's Cover-Up Method

| Partial fraction | Type of test | M | SD | M difference | t | df | Sig. (2-tailed) |
|---|---|---|---|---|---|---|---|
| Distinct linear factor | Pre | 6.10 | 3.71 | 2.50 | 8.35 | 104 | .00 |
| | Post | 8.59 | 3.01 | | | | |
| Repeated linear factor | Pre | 2.49 | 2.27 | 1.63 | 8.94 | 105 | .00 |
| | Post | 4.12 | 2.66 | | | | |
| Irreducible quadratic factor | Pre | 2.82 | 2.54 | 4.02 | 14.61 | 105 | .00 |
| | Post | 6.84 | 3.49 | | | | |

Table 3 indicates students' PFD performance using the undetermined coefficients method. Students had mean pre-test scores varied from 3.30 for repeated linear factor, 4.01 for irreducible quadratic factor, and 4.62 for distinct linear factor, whereas, for post-test scores, they ranged from 4.62 for repeated linear factor, 6.45 for distinct linear factor, and 6.48 for an irreducible quadratic factor. The findings showed a statistically significant increase in the mean test scores when solving the partial decomposition of rational functions with three distinct factors in the denominators. Students achieved the highest score for irreducible quadratic factor followed by distinct linear factor and repeated linear factor.

Table 3. Students' Performance in Pre-Test and Post-Test Using Undetermined Coefficients Method

| Partial fraction | Type of test | M | SD | M difference | t | df | Sig. (2-tailed) |
|---|---|---|---|---|---|---|---|
| Distinct linear factor | Pre | 4.62 | 2.89 | 1.83 | 10.40 | 104 | .00 |
| | Post | 6.45 | 3.26 | | | | |
| Repeated linear factor | Pre | 3.30 | 2.59 | 1.32 | 13.83 | 105 | .00 |
| | Post | 4.62 | 2.75 | | | | |
| Irreducible quadratic factor | Pre | 4.01 | 3.39 | 2.47 | 10.29 | 105 | .00 |
| | Post | 6.48 | 3.56 | | | | |

For distinct linear factor [t(104) = 10.40, $p < .05$; 95% CI (1.48, 2.18)], repeated linear factor [t(105) = 13.83, $p < .05$; 95% CI (1.13, 1.51)], and irreducible quadratic factor [t(105) = 10.29, $p < .05$; 95% CI (1.99, 2.94)], the effect sizes were 0.59, 0.40 and 0.71, respectively. The results of the effect size showed moderate-to-large effects (Cohen, 1988; Kraft, 2020; Mathew, 2019; Nicolas, 2017). In comparison to pre-tests, the findings revealed that using the undetermined coefficients method enhance students' performance in finding partial fraction decomposition in their post-test.

Table 4. Summary of the Analysis of Covariance of the Mean Post-Test Scores Using Different Methods

| Sources of variance | df | Mean Square | F | Sig. |
|---|---|---|---|---|
| Distinct linear factor | | | | |
| Pre-test | 1 | 1001.076 | 197.653 | .000 |
| Methods | 1 | 67.569 | 13.341 | .000 |
| Error | 207 | 5.065 | | |
| Repeated linear factor | | | | |
| Pre-test | 1 | 1401.003 | 577.553 | .000 |
| Methods | 1 | 0.827 | 0.341 | .560 |
| Error | 209 | 2.426 | | |
| Irreducible quadratic factor | | | | |
| Pre-test | 1 | 1206.026 | 179.068 | .000 |
| Methods | 1 | 87.744 | 13.028 | .000 |
| Error | 209 | 6.735 | | |

Table 4 displays the results of an ANCOVA comparison done between the mean post-test scores of the students' partial decomposition of proper rational functions using the improved version of Heaviside's cover-up method and those of the post-test scores obtained using the undetermined coefficients method with distinct linear factor, repeated linear factor, and irreducible quadratic factor, respectively. The analysis showed a significant difference between the performance in the partial decomposition of proper rational functions using the two methods with respect to distinct linear factor [F(1, 207) = 13.34, $p < 0.05$; eta-squared = 0.06; Levene’s test: $p$-value = 0.17], and irreducible quadratic factor [F(1, 209) = 13.03 , $p < 0.05$; eta-squared = 0.06; Levene’s test: $p$-value = 0.09]. However, with respect to the findings for repeated linear factor, there was no significant difference between the performance in the partial decomposition of proper rational functions using the two methods [F(1, 209) = 0.83 , $p > 0.05$; eta-squared = 0.00; Levene’s test: $p$-value = 0.00]. The mean post-test scores obtained using the Heaviside's cover-up method for distinct linear factor were 2.14 higher than the mean post-test scores obtained using the undetermined coefficients method. Similarly, for the irreducible quadratic factor, the mean post-test scores obtained by Heaviside's cover-up method were 0.36 higher than the mean post-test scores obtained by the undetermined coefficients method. Conversely, students who used Heaviside's cover-up method recorded 0.51 lower on the mean post-test scores of repeated linear factors than those students who used the undetermined coefficients method. Subsequently, the eta-squared statistics determines the difference in effect sizes between the two methods being compared whilst controlling for students' pre-test scores. The eta-squared (0.06) value showed a moderate effect size (Cohen, 1988; Kraft, 2020). This value indicated that the students who used Heaviside's cover-up method outperformed students who used the undetermined coefficients method in the partial fraction decomposition of proper rational functions concerning distinct linear factor and irreducible quadratic factor in the denominators. However, students displayed similar achievements in both methods for the partial fraction decomposition of proper rational functions with repeated linear factor in the denominator.

## 5. DISCUSSION AND CONCLUSION

This study aims to explore the effectiveness of applying the improved version of Heaviside's cover-up method and undetermined coefficients method in performing PFD, specifically of the proper rational functions. The main difference between applying the two methods is that students who use the undetermined coefficients method solve the system linear equations via algebra approach to find PFD coefficients, while students who use the improved version of the Heaviside's cover-up method evaluate PFD coefficients by performing substitution process in simple polynomial division functions. The results shown in Table 4 indicate that students who used the improved version of Heaviside's cover-up method outperformed those who used the undetermined coefficients method in the partial fraction decomposition of proper rational functions for distinct linear factor and irreducible quadratic factor in the denominators. However, the performance of using both methods is the insignificant difference for solving partial fraction decomposition of proper rational functions concerning repeated linear factors in the denominator. It may be ascribed to students' lack of mastery in applying the methods and their difficulties working with three partial fractions due to the lengthy, complicated, and inconvenient computations (Wang, 2007 cited in Jong & Kuan, 2020; Man, 2012). The finding of this study is consistent with the previous studies which postulated that students had a lack of proficiency in concepts of computing fractions (e.g., Bentley & Bossé, 2018; Gabriel et al., 2013; Titus, 2010; Steen, 2007) which resulted in low academic performance of the students (Bentley & Bossé, 2018; Kor et al., 2019; Jong & Kuan, 2020). In addition, some researchers purported that the major obstacles in students' failure to solve the system linear equations were weak in algebraic conceptual knowledge (Taban & Cadorna, 2018; Jupri et al., 2014; Hoch & Dreyfus, 2010; Jong & Kuan, 2020), inability to recognize the algebraic structures (Novotna & Hoch, 2008; Taban & Cadorna, 2018), and a lack of manipulating algebraic expressions skills (Booth el at., 2014; Ashlock, 2010). The difficulties mentioned above faced by students are also reflected in students' mean post-test scores employing Heaviside's cover-up method as the scores obtained are somewhat higher than their mean post-test scores using the undetermined coefficients method, as shown in Tables 2 and 3.

The discussions on the findings obtained in this study have indicated the promising potential in raising the students' performance, particularly in terms of computing competence and getting accurate solutions when solving coefficient partial fraction decomposition problems using the improved version of the Heaviside's cover-up method in view of the progress made in the students' performance after having learnt this method. Furthermore, it is believed that the improved version of the Heaviside cover-up method could address the identified common mistakes made when applying the undetermined coefficients method, namely failure to solve the system linear equations and a lack of proficiency in concepts of computing fractions. In conclusion, the improved version of Heaviside's cover-up method is highly recommended to teach undergraduates in the elementary integral calculus courses. Nevertheless, this recommendation is restricted in its generalization to other contexts, both nationally and internationally, since samples were drawn from only two faculties in a public university in Sarawak. Despite this limitation, this study provides insights into the impact of applying two different PFD methods on students' academic performance. The researchers of this study hope to create awareness among the instructors in their choices to teach students PFD with the belief that different instructional methods may drive students to explore and understand partial fractions to bring out the best in their PFD performance. Future research should examine the replication of the findings on different PFD methods at other universities with larger sample sizes. Such studies/research could serve as an instructional development to the small research group and instructors in this area concerning PFD methods on the academic performance of students.

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## APPENDIX

### *Appendix 1: Student Assessment*

UNIVERSITI TEKNOLOGI MARA

**ASSESSMENT**

| | | |
|---|---|---|
| **COURSE** | : | **CALCULUS II** |
| **COURSE CODE** | : | **MAT238/235** |
| **DATE** | : | **FEBRUARY 2020/2021** |
| **DURATION** | : | **1 Hour** |

**INSTRUCTIONS TO CANDIDATES**

1. This examination consists of **THREE** questions.
2. Answer **ALL** questions in English. Show all the necessary working solutions.
3. Every sheet of paper **MUST** include your **name, ID, group, lecturer's name, and date**.

Use **the undetermined coefficients method** and **the improved Heaviside's cover-up method** to decompose the following proper rational functions.

**Question 1**

Find the partial fraction expansion of the proper rational function for distinct linear polynomial denominator, $\mathrm{G}(x)=\dfrac{1-2x}{6x^2+x-1}$.

**Question 2**

Find the partial fraction expansion of the proper rational function for repeated linear polynomial denominator, $\mathrm{G}(x)=\dfrac{3+4x}{(x)(x+5)^2}$.

**Question 3**

Find the partial fraction expansion of the proper rational function for irreducible quadratic polynomial denominator, $\mathrm{G}(x)=\dfrac{2-5x}{(x)(x^2+16)}$.

## AUTHOR BIOGRAPHIES

**Jong Li Ling** is a lecturer at the Department of Mathematical Sciences, Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Cawangan Sarawak. Her main research interest is in the areas of Mathematics and Statistics.

**Kuan Siaw Vui** received her Ph.D. in Management from Universiti Sains Malaysia. She is a senior lecturer at the Faculty of Business Management, Universiti Teknologi MARA, Cawangan Sarawak. Her main research interest is in the areas of Leadership, Mentorship, Job Performance and Occupational Success.